A JUMP OPTION PRICING MODEL WITH TRANSACTION COSTS

Averbuj, Corina
1 Introduction

Since Black and Scholes’s paper ([2]) presents a formula to pricing option, there has been an increasing interest on problems arising in Financial Mathematics and in particular on derivatives pricing. The standard approach to this problem leads to the study of equations of parabolic type.

In the standard Black-Scholes model, a basic assumption is that the stock price follows a geometric Brownian motion through time.

Empirical studies exhibit systematic biases on stock price series. Several authors as Merton ([12]) propose an option pricing model that explicitly admits jumps in the underlying security return process, they consider the stock price dynamic with two component:

a) The continuous component represented by a Wiener process and
b) The jump component is represented by a ”Poisson-driven” process.

The stock price return can be formally written as

\[
\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dZ + dq
\]

where \(\alpha\) is the instantaneous expected return on the stock; \(\sigma^2\) is the instantaneous variance conditional on no arrival of new information, \(\lambda\) is the mean number of arrivals per unit of time, \(k = E(Y - 1), Y \geq 0\) where \((Y - 1)\) is the random variable percentage change in the stock price and \(E\) is the expectation operator over the random variable \(Y\) (for details see [12],[13]).

In [12],[13] an option pricing formula is derived and the following partial integro-differential equation (PIDE) on the variable \(\tau\) and \(S\) is obtained:

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + (\alpha - \lambda k) S \frac{\partial F}{\partial S} - \frac{\partial F}{\partial \tau} - \lambda E \{F(SY, \tau) - F(S, \tau)\} = 0 \quad (1)
\]
Here \( \tau \) is time until option expiration and \( g(S, \tau) \) is the equilibrium, instantaneous rate of return on the option \( F(S, \tau) \).

If the jump components represent non-systematic risk reflecting the arrival of new information specific to the firm or its industry, they will be uncorrelated with the market, then the expect return on these securities must equal the riskless rate and in the (1) \( \alpha \) and \( g(S_1, \ldots, S_N, \tau) \) will be changed by \( r \). (for details see [12],[13])

In order to proposed a model that fit well with empirical investigations, Kou ([9]) supposes that the logarithm of jump sizes having a double exponential distribution and the jump times corresponding to the event times of a Poisson process. Press ([14]) suposse that the jump sizes having a Lognormal distribution.

Merton ([13]) suposse that the jump sizes are Normal and Kim Jig-M Jung ([8]) suposse that jump sizes having a Poisson distribution

One of the classic assumption with the Black-Scholes model’s resolution is that the investor’s portfolio revalues in a continuous form.

This dynamic implies transaction costs, due to the buy/sell of necessary stocks to maintain the portfolio’s equilibrium. Black-Scholes models which include transaction costs were studied by many authors ([15],[11]).

In this work we suppose that transaction costs behave as a nonincreasing linear function, \( h(x) = ax - bx \), \( a, b > 0 \), depending on the trading stocks need to hedge the portfolio that replicates the contingent claim.

In the next section we study a generalization of PIDE (1) including transaction costs to the model.

## 2 Derivation of Valuation Problem

As Merton ([12]), we assume the stock price return follows the SDE

\[
\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dZ + dQ
\]

rewritten as

\[
\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dZ \text{ if the Poisson process does not occur} \\
= (\alpha - \lambda k)dt + \sigma dZ + (Y - 1) \text{, if the Poisson process occurs}
\]

\( dZ \) and \( dQ \) are assumed to be independent, \( k = E(Y - 1) \), where \( Y - 1 \) is the random variable percentage change in the stock price, \( \lambda \) is the mean number of arrival per unit time.

Following the idea of Leland ([11]), if the value of the option is denoted by \( V(S, t) \), where \( S \) is the value of the underlying asset, for \( \Pi = V - \triangle S \) we have:

\[
\delta \Pi = \delta V - \triangle \delta S - (a - (b \mid \nu \rangle S \mid \nu \rangle) S \mid \nu \rangle = \delta V - \triangle \delta S - (a - b \mid \nu \rangle \delta S \mid \nu \rangle - (a - b \mid \nu \rangle S \mid \nu \rangle)
\]

(2)
where \( \nu \) is the number of shares of the asset which are traded in order to maintain the equilibrium of the portfolio in the periodo \((t, t + \delta t)\).

Using Ito’s Lemma for the continuous part and analogous lemma for the jump part ([10]) we have

\[
\delta V = \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S (r - \lambda k) \frac{\partial V}{\partial S} \right] \delta t + \sigma \frac{\partial V}{\partial S} \delta Z + [V(S_{y}, t) - V(S, t)] \delta Q
\]

and

\[
\delta V - \Delta \delta S = \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S (r - \lambda k) \frac{\partial V}{\partial S} \right] \delta t
\]

\[
+ \sigma \frac{\partial V}{\partial S} \delta Z + [V(S_{y}, t) - V(S, t)] \delta Q
\]

\[
- \Delta [S (r - \lambda k) \delta t + \sigma S \delta Z + S \delta Q]
\]

\[
- [(a - b |\nu|) \delta S |\nu| - [(a - b |\nu|) S |\nu|]
\]

If we choose \( \Delta = \frac{\partial V}{\partial S} (S, t) \) as the number of shares to maintain in the instant \( t \).

Then, the number of shares of the asset which are traded in order to maintain the equilibrium of the portfolio in the periodo \((t, t + \delta t)\) is:

\[
\nu = \frac{\partial V}{\partial S} (S + \delta S, t + \delta t) - \frac{\partial V}{\partial S} (S, t)
\]

By Ito’s Lemma,

\[
\nu \simeq \frac{\partial^2 V}{\partial S^2} \delta S + \left[ \frac{\partial V}{\partial S} (S_{y}, t) - \frac{\partial V}{\partial S} (S, t) \right] \delta Q + O(\delta t)
\]

Applying the expected value, we obtain that

\[
E \left\{ \left[ \frac{\partial V}{\partial S} (S_{y}, t) - \frac{\partial V}{\partial S} (S, t) \right] \delta Q \right\} = \lambda \delta t E \left\{ \left[ \frac{\partial V}{\partial S} (S_{y}, t) - \frac{\partial V}{\partial S} (S, t) \right] \right\} \simeq O(\delta t)
\]

As \( \delta S = \sigma S \phi \sqrt{\delta t} + O(\delta t) \), with \( \phi \sim N(0, 1) \), we have

\[
\nu \simeq \frac{\partial^2 V}{\partial S^2} (S, t) \sigma S \phi \sqrt{\delta t}
\]

and \( E \{ \delta \Pi \} = E \{ \delta V - \Delta \delta S \} - E \{ [(a - b |\nu|) S |\nu|] \}
\]

Then,

\[
E [(a - b |\nu|) S |\nu|] = E [a S |\nu| - b S \nu^2] = E(a S |\nu|) - E(b S \nu^2) = \frac{3}{2}
\]

(3)
Using that $E(|\phi|) = \sqrt{\frac{2\alpha}{\pi}}$, $E(\phi^2) = \delta t$ and (3) we find that

\[ E[(a - b|\nu|)S|\nu|] = \left| \frac{\partial}{\partial S} \right| \sigma S^2 \sqrt{\frac{2}{\pi}} \sqrt{\delta t} a - b S^3 \left( \frac{\partial}{\partial S} \right)^2 \sigma^2 \delta t \]

Hence we obtain the equation

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \left( \frac{\partial^2 V}{\partial S^2} \right) - a \left| \frac{\partial V}{\partial S} \right| \sigma S^2 \sqrt{\frac{2}{\pi \delta t}} + \left( \frac{\partial^2 V}{\partial S^2} \right)^2 b S^3 \sigma^2 + (r - \lambda k) S \frac{\partial V}{\partial S} - r V + \lambda E [V(Sy, t) - V(S, t)] = 0 \]  

(4)

Assuming that $a$ is small enough, we have that $\bar{\sigma}^2 = \sigma^2 \left( 1 - \frac{a^2}{\sigma^2} \sqrt{\frac{2}{\pi \delta t}} \right) > 0$.

We know that classical solution of Black-Scholes’ equation is

\[ C(S, t) = SN(d_1) - Ke^{-r(T-t)} N(d_2) \]

and

\[ \frac{\partial C}{\partial S}(S, t) = N(d_1) \]

\[ \frac{\partial^2 C}{\partial S^2}(S, t) = \frac{e^{-d_1^2}}{S \sigma \sqrt{\delta t}} > 0 \]

Because the european call option is a convex function respect of $S$, we assume $\frac{\partial^2 V}{\partial S^2} > 0$.

In the next section we study the stationary problem for (4) under the Dirichlet boundary conditions.

2.1 The Stationary Problem

In this section we apply the Schaefer’s fixed point theorem to find convex solutions of the problem

\[ \begin{cases} 
\frac{1}{2} \sigma^2 S^2 \left( \frac{\partial^2 V}{\partial S^2} \right) + \left( \frac{\partial V}{\partial S} \right)^2 b S^3 \sigma^2 + (r - \lambda k) S \frac{\partial V}{\partial S} - r V + \lambda E [V(Sy, t) - V(S, t)] = 0 \\
V(c) = V_c, \quad V(d) = V_d \end{cases} \]  

(2.1)

or equivalently

\[ V'' + H(S, V, V') = 0 \quad V(c) = V_c \quad V(d) = V_d \]

where

\[ H(S, V, V') = \frac{1}{2} \sigma^2 S^2 - \sqrt{\left[ \frac{S^4 \bar{\sigma}^4}{4} + 4bS^3 \sigma^2 |S(r - \lambda k)V' - r V + \lambda G(V)| \right]} \]

and $G : H^1(c, d) \to H^1(c, d)$ is the operator given by $G(V)(S) = \int f(y) [V(Sy) - V(S)] dy$.
such that \( d(r - \lambda k)V'(d) - rV(d) + \lambda G(V)(d) < 0 \)

If \((r - \lambda k) > 0\) and \(E(yV'(cy)) > (k+1)\|V'\|_{L^2} \), then \((S(r - \lambda k)V' - rV + \lambda G(V))' > 0\) (see Appendix A) and \(H\) is well defined.

**Remark 1** A simple computation shows that \(H\) satisfies the following conditions:

\[
|H(S, V, X) - H(S, U, X)| \leq Kr\|V - U\| + K\lambda \|G(V) - G(U)\|_{\infty}
\]

and

\[
|H(S, X, V') - H(S, X, U')| \leq K'\|V' - U'\|
\]

where \(K = \frac{2}{\sigma^2 c^2}\) and \(K' = \frac{2r}{\sigma^2 c}\). We have assume that \(K' < \frac{\pi}{d - c}\).

**Theorem 1** Assuming that \((r - \lambda k)V'(c) - rV(c) + \lambda G(V)(c) < d(r - \lambda k)V'(d) - rV(d) + \lambda G(V)(d)\) hold, (2.1) admits a convex solution.

**Proof.** Let \(\beta > 0\) be large enough and given \(V \in H^1(c, d)\), we may define the operator \(K : H^1(c, d) \rightarrow H^1(c, d)\) by \(K(V) = V\) where \(V\) is the unique solution of the linear problem ([6])

\[
\begin{cases}
V'' - \beta V = -\beta V - H(S, V, V') \\
V(c) = v_c, \quad V(d) = v_d
\end{cases}
\]

In order to prove that \(K\) is a continuous operator, we take \(u \in H^1(c, d)\) such that \(K(u) = u\), then

\[
(V - u)'' - \beta (V - u) = \beta (V - u) + H(S, u, u') - H(S, V, V')
\]

As

\[
\begin{align*}
\left| H(S, u, u') - H(S, V, V') \right| & \leq \left| H(S, u, u') - H(S, u, V') \right| + \left| H(S, u, V') - H(S, V, V') \right| \\
& \leq K' \left| u - V \right| + K \left| u - V \right| + \lambda K \left\| G(u) - G(V) \right\|_{\infty} \\
& \leq K' \left| u - V \right| + K \left| u - V \right| + \lambda K \left\| f \right\|_{\infty} + \lambda K \left\| V - u \right\|_{L^2(c, d)}
\end{align*}
\]

By regularity conditions (Theo. 6.3 [5]), we have

\[
\|V - u\|_{H^2(c, d)} \leq C \left\| \beta (V - u) + H(S, u, u') - H(S, V, V') \right\|_{L^2(c, d)}
\]

\[
\|V - u\|_{H^2(c, d)} \leq C \left\| V - u \right\|_{H^1(c, d)} \tag{5}
\]

for some constant \(C > 0\), and then \(K\) is continuous.
Moreover $K$ is compact, because if $\nabla_n - v_L$ is bounded in $H^1_0(c,d)$ where $v_L$ is the line through the points $(c;v_c)$ and $(d;v_d)$. Using (5) we have that $K \nabla_n - K v_L$ is bounded in $H^2(c,d)$. By the compact imbedding $H^2(c,d) \hookrightarrow C^1([c,d])$ there exists a subsequence that strongly convergent in $H^1_0(c,d)$.

Finally, we shall show that 
\[ \{ v \in H^1(c,d) \text{ such that } v = \zeta K v \text{ for some } \zeta \in [0,1] \} \]
is bounded in $H^1(c,d)$

Let $\zeta K(u) = u$ for some $\zeta \in [0,1]$ and $v_L$ is the line through the points $(c;\zeta u_c)$ and $(d;\zeta u_d)$

\[
(u - v_L)'' - \beta (u - v_L) = \beta u \zeta - \zeta H(S, u, u') + \beta v_L
\]

= 

\[-\beta \zeta (u - v_L) + \beta (1 - \zeta) v_L - \zeta \vartheta(S)(u - v_L) + \phi(S)(u - v_L)' + H(S, v_L, v_L')\]

where \( \vartheta(S) = \frac{H(S, u, u') - H(S, v_L, v_L')}{u - v_L} \in L^\infty(c,d) \) and \( \phi(S) = \frac{H(S, v_L, v_L') - H(S, v_L, v_L')}{u - v_L} \in L^\infty(c,d) \)

Then

\[- \int (u' - v_L')^2 + \beta (\zeta - 1)(u - v_L)^2 = \int \beta (1 - \zeta) v_L(u - v_L) - \zeta \vartheta(S)(u - v_L)
\]

\[+ \int \phi(S)(u - v_L)' + H(S, v_L, v_L')(u - v_L)\]

\[||u' - v_L'||_2 + \beta (1 - \zeta) ||u - v_L||_2^2 \leq \beta (1 - \zeta) ||u||_\infty ||u - v_L||_2 + \zeta ||\vartheta||_\infty ||u - v_L||_2^2
\]

\[+ \zeta ||\phi||_\infty ||u - v_L||_2 ||u' - v_L'||_2 + \zeta ||H||_\infty ||u - v_L||_2^2\]

Using Poincaré’s Theorem \( ||u - v_L||_2 \leq \frac{d-c}{\pi} ||u' - v_L'||_2 \) and taking $\beta > 0$ sufficiently large we have \( ||u - v_L||_{H^1_0(c,d)} \leq C ||u - v_L||_{H^1_0(c,d)} \)

Thus \( ||u - v_L||_{H^1_0(c,d)} \leq C \) for constant $C$ independent of $\zeta$ (see appendix

B) \]

**Example 1** In order to interpret the parameter $\lambda$ we can consider the derivative \( F(S_y, T) = \) \[
\begin{align*}
0 & \text{ if } dq = 1 \\
1 & \text{ if } dq = 0
\end{align*}
\]

Because of $dF$ and $dS$ are statiscally independent we see that $F$ satisfy the following ordinary differential equation:

\[
\frac{\partial F}{\partial T} - (r + \lambda) F = 0
\]

The solution is given by $F(t) = e^{-(r+\lambda)(T-t)}$ so $\lambda$ is the hazard rate of the Poisson process.
3 Appendix A

Let $\eta(S) = E(yV'(Sy))$ a simply computation shows that $\eta(S)$ is a nondecreasing function, so

If $E(yV'(cy)) > (k + 1) \|V\|_{L^2}$, then $E(yV'(cy)) \geq E(yV'(cy)) > (k + 1)V'(S)$

$(S(r - \lambda k)V' - rV + \lambda G(V))' = ((r - \lambda k)V' + S(r - \lambda k)V - rV' + \lambda E(yV'(Sy) - \lambda V' > S(r - \lambda k)V' > 0

4 Appendix B

$$\|u' - v'_L\|_2^2 + \beta(1 - \zeta) \|u - v_L\|_2^2 \leq \beta(1 - \zeta) \|v_L\|_{\infty} \|u - v_L\|_2 + \zeta \|\phi\|_{\infty} \|u - v_L\|_2 + \zeta \|H\|_{\infty} \|u - v_L\|_2$$

$$(1 - \zeta \|\phi\|_{\infty} \frac{d - c}{\pi} \|u' - v'_L\|_2 + (\beta(1 - \zeta) - \zeta \|\theta\|_{\infty} - \zeta \|\phi\|_{\infty}) \|u - v_L\|_2 \leq \beta(1 - \zeta) \|v_L\|_{\infty} \frac{d - c}{\pi} \|u' - v'_L\|_2$$

And

$$A \|u - v_L\|_{H^1(c,d)}^2 \leq \beta \|v_L\|_{\infty} \frac{d - c}{\pi} \|u' - v'_L\|_2$$

Thus $\|u - v_L\|_{H^1(c,d)}^2 \leq C$ where for fixed $\beta > 0$ big enough and $C = \frac{\beta \|v_L\|_{\infty} \frac{d - c}{A \pi}}{d - c}$

References


