Group strategy-proof social choice functions

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THE FRAMEWORK

- Let A be the set of **alternatives** and $N = \{1, 2, .., n\}$ be the set of **agents** (with $n \geqslant 2$)

- Agents' **preferences** are complete and transitive binary relations on A. Let $\Re$ be the set of all such relations.

- $\mathcal{R}_i \subset \Re$ denotes the set of admissible preferences for agent i (not necessarily the same for any agent)

- $R_i \in \mathcal{R}_i$, P_i and I_i the strict and indifference part, respectively

- Let $L(R_i, x) = \{y \in A : xR_iy\}$ denote the lower contour set of R_i at x

- Similarly, $\overline{L}(R_i, x) = \{y \in A : xP_iy\}$ denotes the strict lower contour set of R_i at x

THE FRAMEWORK (cont.)

- A **Social Choice Function** is a function $f : \times_{i \in N} \mathcal{R}_i \longrightarrow A$

- As usual, $\times_{i \in N} \mathcal{R}_i$ will be called the domain of f

- Abusing language, we'll call any set $\times_{i \in N} \mathcal{R}_i$ a domain, even when it is not referred to a particular function. In particular, $\Re^n$ will be called the universal domain

INDIVIDUAL AND GROUP STRATEGY-PROOFNESS

- A coalition C **can manipulate** f on $\times_{i \in N} \mathcal{R}_i$ at R_N if there exists $R'_C \in \times_{j \in C} \mathcal{R}_j$, where $R'_j \neq R_j$ for any $j \in C$ and $f(R'_C, R_{-C}) P_j f(R_N)$ for any $j \in C$

- An SCF f is **Group Strategy-Proof** (GSP) on $\times_{i \in N} \mathcal{R}_i$ if no coalition $C \subseteq N$ can manipulate f on $\times_{i \in N} \mathcal{R}_i$ at any R_N

- An SCF f is **Strategy-Proof** (SP) on $\times_{i \in N} \mathcal{R}_i$ if no singleton $\{i\}$ can manipulate f on $\times_{i \in N} \mathcal{R}_i$ at any R_N

- An SCF f is k-**Group Strategy-Proof** (k-GSP) on $\times_{i \in N} \mathcal{R}_i$ if no coalition $C \subseteq N$ with $\#C \leq k$ can manipulate f on $\times_{i \in N} \mathcal{R}_i$ at R_N

Heuristically we can argue that the threat of manipulation may decrease as k increases due to coordination costs

GIBBARD-SATTERHWAITE THEOREM

Theorem

Let f be a voting scheme on the universal domain whose range contains more than two alternatives. Then f is either dictatorial or manipulable.

ONE WAY OUT: RESTRICTED PREFERENCE DOMAINS. THE CASE OF SINGLE-PEAKED PREFERENCES

- Finite set of alternatives linearly ordered according to some criterion.
- Preference of agents over alternatives is single-peaked.

- Each agent has a single preferred alternative $\tau(R_i)$
- If alternative z is between x and $\tau(R_i)$, then z is preferred to x

- Consider the case where the number of alternatives is finite, and identify them with the integers in an interval $[a, b] = \{a, a+1, ..., b\} = A$ (Moulin(1980a)).

THE CASE OF LINEARLY ORDERED SETS OF ALTERNATIVES. POSSIBILITY RESULTS

- **Example 1** There are three agents. Allow each one to vote for her preferred alternative. Choose the median of the three voters.
- **Example 2** There are two agents. We fix an alternative p in $[a, b]$. Agents are asked to vote for their best alternatives, and the median of p, 1 and 2 is the outcome.
- **Example 3** For any number of agents, ask each one for their preferred alternative and choose the smallest.
- Notice that all three rules are anonymous and strategy-proof.

THE CASE OF LINEARLY ORDERED SETS OF ALTERNATIVES: A CHARACTERIZATION RESULT

Theorem

Theorem (Moulin, 1980a) An anonymous social choice function on profiles of single-peaked preferences over a linearly ordered set is strategy-proof if and only if there exist $n+1$ *points* $p_1, ..., p_{n+1}$ *in* A *(called the phantom voters), such that, for all profiles,*

$$f(R_1, ..., R_n) = med(p_1, ..., p_{n+1}; \tau(R_1), ..., \tau(R_n))$$

Remark: Moreover, all these rules are also **Group Strategy-proof.**

ANOTHER SPECIAL CASE: VOTING BY COMMITTEES

- (Barberà, Sonnennschein, and Zhou (1991)). Consider a club composed of N members, who are facing the possibility of choosing new members out of the set of K candidates. Are there any strategy-proof rules the club can use?

Yes, there are, if preferences are separable. In particular, voting by quota rules are strategy-proof, anonymous and neutral.

VOTING BY COMMITTEES ON SEPARABLE PREFERENCES

- Let $N = \{1, 2\}$, two candidates a, b can be elected: $A = \{\varnothing, a, b, \{a, b\}\}$
- f voting by quota 1 is strategy-proof.

Admissible individual preferences							
R^1	R^2	$\mathbf{R}^3$	R^4	$\mathbf{R}^5$	R^6	R^7	R^8
$\varnothing$	$\varnothing$	a	a	b	b	$\{a, b\}$	$\{a, b\}$
a	b	$\varnothing$	$\{a, b\}$	$\varnothing$	$\{a, b\}$	a	b
b	a	$\{\mathbf{a}, \mathbf{b}\}$	$\varnothing$	$\{\mathbf{a}, \mathbf{b}\}$	$\varnothing$	b	a
$\{a, b\}$	$\{a, b\}$	b	b	a	a	$\varnothing$	$\varnothing$

- However, **it is not strategy-proof !**
- If $R_N = (R^3, R^5)$ then $f(R_N) = \{a, b\}$
- If $R'_N = (R^1, R^2)$ then $f(R'_N) = \varnothing$
- N manipulates f at R_N via R'_N

THE CONNECTIONS BETWEEN INDIVIDUAL AND GROUP STRATEGY PROOFNESS

Our starting Remark

- Sometimes there are strategy-proof rules that are also group strategy-proof
(Ex: majority on single-peaked preferences)
- Sometimes not
(Ex: voting by quota 1 on separable preferences)

Our Main Question

Are there domains where the coincidence between strategy-proofness and group strategy-proofness is not a matter of one specific rule, but would occur for any rule that can be defined on them?

Our Answer will be YES:

We first identify a basic condition on preferences having the property that if satisfied, all strategy-proof rules on that domain will be also group strategy-proof. Then, we weaken this condition in several directions and get other related results

RELATED LITERATURE

Similar subject different question

- Pattanaik (1978), Dasgupta, Hammond and Maskin (1979), and Green and Laffont (1979)

- Barberà (1979), Barberà, Sonnenschein, and Zhou (1991) and Serizawa (2006)

- Barberà and Jackson (1995), Moulin (1999) and Pápai (2000)

- Peleg (1998) and Peleg and Sudholter (1999)

Same question, special case

- Le Breton and Zaporozhets (2008): Each agent admissible domain of preferences is the same. They propose a richness condition on individual domains to guarantee the equivalence between strategy-proofness and group strategy-proofness. This condition, in contrast to ours, requires domains to be "sufficiently large"

SOME EXAMPLES

- Domains where strategy-proof rules are group strategy-proof because our basic condition holds:

- Any subset of Single-peaked preferences with respect to a given order of alternatives
- Any subset of Single-dipped preferences with respect to a given order of alternatives

- Domains where strategy-proof rules are group strategy-proof because a weaker but still sufficient condition holds:

- Universal domain
- Domains where some agents have single-peaked preferences and others have single-dipped preferences with respect to the same order of alternatives (if they are rich enough)

- A domain where a strategy-proof rule may exist that is not group strategy-proof:

- Separable preferences

THE SEQUENTIAL INCLUSION CONDITION

Definition

Given $R_N \in \times_{i \in N} \mathcal{R}_i$ and $y, z \in A$, define a binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z) \equiv \{i \in N : yP_i z\}$ such that $i \succsim (R_N; y, z) j$ if $L(R_i, z) \subset \overline{L}(R_j, y)$

Note that $\succsim (R_N; y, z)$ is reflexive but not necessarily complete

Definition

A preference profile $R_N \in \times_{i \in N} \mathcal{R}_i$ satisfies **sequential inclusion** if for any pair $y, z \in A$ the binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete and acyclic

THE SEQUENTIAL INCLUSION CONDITION (cont.)

An equivalent condition:

Lemma

$R_N \in \times_{i \in N} \mathcal{R}_i$ satisfies sequential inclusion if and only if for any pair $y, z \in A$ there exists a linear order of agents of $S(R_N; y, z)$, say $1 < 2 < ... < s$, such that for all sequences $z_1, z_2, ..., z_{s-1}$ where $z_1 = z$ and $z_i \in L(R_{i-1}, z_{i-1})$, for any $i = 2, ..., s-1$, we have that $[L(R_j, z_j) \subset \overline{L}(R_h, y)$ for all h, $j + 1 \leq h \leq s]$ for all $j = 1, ..., s-1$

Main result:

Theorem

Let $\times_{i \in N} \mathcal{R}_i$ be a domain such that any preference profile $R_N \in \times_{i \in N} R_i$ satisfies the sequential inclusion condition. Then, any SP rule on that domain is also GSP

TWO IMPORTANT FEATURES

This condition has **Two Important Features**

1. It applies to each preference profile individually.

Therefore, if it is satisfied for a domain it is also satisfied by any subdomain.

2. It may be partially satisfied:

- For a subset of agents

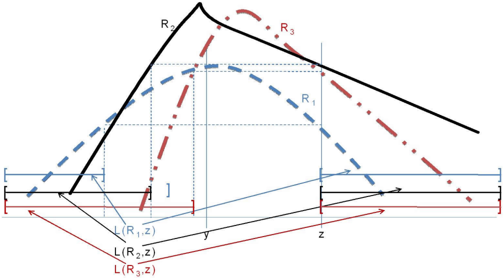
This allows us to answer: *When is strategy-proofness equivalent to k-group strategy-proofness?*

- For a subset of alternatives

This allows us to answer: *When does the equivalence hold if the range is somewhat rectricted?*

EXAMPLE: SINGLE-PEAKED PREFERENCES

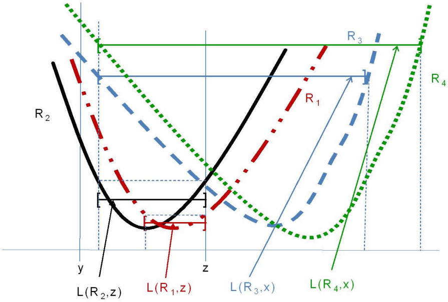
- Fix R_N and $y < z$ (see the graphic below)



- Agents in $S(R_N; y, z)$ (1,2,3 in the graphic) can be always ordered according to the increasing order of lower contour sets at z: w.l.o.g., say $L(R_1, z) \subset ... \subset L(R_s, z)$
- Clearly, $1 \succsim 2, 3, ..., s$; $2 \succsim 3, ..., s$; ...; $s - 1 \succsim s$ since yP_iz for any $i \in S(R_N; y, z)$
- Thus $\succsim (R_N; y, z)$ is complete and acyclic

EXAMPLE: SINGLE-DIPPED PREFERENCES

- Fix R_N and $y < z$ as follows:



- $S = S_1 \cup S_2$. Define the order of S_1: 1¡2 since $L(R_1, z) \subset L(R_2, z)$
- Let $x = \min L(R_2, z)$, order of S_2: 3¡4 since $L(R_3, x) \subset L(R_4, x)$
- $L(R_1, z) \subset \overline{L}(R_{h>1}, y)$; for any $z_2 \in L(R_1, z)$, $L(R_2, z_2) \subset \overline{L}(R_{h>2}, y)$, for any $z_3 \in L(R_2, z_2)$, $L(R_3, z_3) \subset \overline{L}(R_{h>3}, y)$

EXAMPLE: SEPARABLE PREFERENCES VIOLATES SEQUENTIAL INCLUSION

- Two candidates a, b can be elected: $A = \{\varnothing, a, b, \{a, b\}\}$

Admissible individual preferences							
R^1	R^2	$\mathbf{R}^3$	R^4	$\mathbf{R}^5$	R^6	R^7	R^8
$\varnothing$	$\varnothing$	$\mathbf{a}$	a	$\mathbf{b}$	b	$\{a, b\}$	$\{a, b\}$
a	b	$\varnothing$	$\{a, b\}$	$\varnothing$	$\{a, b\}$	a	b
b	a	$\{\mathbf{a}, \mathbf{b}\}$	$\varnothing$	$\{\mathbf{a}, \mathbf{b}\}$	$\varnothing$	b	a
$\{a, b\}$	$\{a, b\}$	$\mathbf{b}$	b	$\mathbf{a}$	a	$\varnothing$	$\varnothing$

- W.l.o.g. let $N = \{1, 2\}$. Let $R_N = (R^3, R^5)$ and $y = \varnothing$, $z = \{a, b\}$
- $S(R_N; y, z) = \{1, 2\}$ since $\varnothing P^3 \{a, b\}$ and $\varnothing P^5 \{a, b\}$
- But $L(R^3, \{a, b\}) \not\subseteq \overline{L}(R^5, \varnothing)$ and $L(R^5, \{a, b\}) \not\subseteq \overline{L}(R^3, \varnothing)$: R_N does not satisfy SI since $\succsim (R_N; y, z)$ is not complete

SPECIAL RESULTS: (1) Two or three alternatives

The two or three alternatives case:

Lemma

Let $\#A \leq 3$. Then, any profile of preferences $R_N \in \times_{i \in N} \mathcal{R}_i$ satisfies the sequential inclusion condition and any strategy-proof social choice function on $\times_{i \in N} \mathcal{R}_i$ is also group strategy-proof

SPECIAL RESULTS: (2) The richness of the domain

Large domains like the universal can be encompassed

Definition

For preferences $R_i, R'_i \in \mathcal{R}_i$ and alternative $x \in A$, R'_i is a strict monotonic transformation of R_i at x if R'_i is such that for all $y \in A \backslash \{x\}$ such that xR_iy, xP'_iy

Definition

A domain $\times_{i \in N}\mathcal{R}_i$ satisfies **indirect sequential inclusion** if, for all profiles $R_N \in \times_{i \in N}\mathcal{R}_i$, either (a) the profile R_N satisfies sequential inclusion, or else (b) for each pair $y, z \in A$ there exists $R'_N \in \times_{i \in N}\mathcal{R}_i$ where $R'_{N \backslash S} = R_{N \backslash S}$ and $S = \{i \in N : yP_iz\}$, such that
(1) for any $j \in S$, R'_j is a strict monotonic transformation of R_j at z,
(2) for any $i \in S$, yP'_iz and
(3) R'_N satisfies sequential inclusion for y, z

SPECIAL RESULTS: (3) k-group strategy-proofness

k-GSP: bounding the size of manipulators

We have already noticed that sometimes rules are manipulable by groups of certain sizes but not for others

Definition

A social choice function f is k-group strategy-proof on $\times_{i \in N} \mathcal{R}_i$ if for any $R_N \in \times_{i \in N} \mathcal{R}_i$, there is no coalition $C \subseteq N$ with $\#C \leq k$ that manipulates f on $\times_{i \in N} \mathcal{R}_i$ at R_N

Note that k-group strategy-proof implies l-group strategy-proof for $l < k$. The converse is not true

New question: Are there domains where the coincidence between strategy-proofness and k-group strategy-proofness would occur for any rule that can be defined on them?

SPECIAL RESULTS: (3) k-group strategy-proofness (cont.)

Our answer: **New condition**

Definition

A preference profile $R_N \in \times_{i \in N} \mathcal{R}_i$ satisfies k-size sequential inclusion if for any pair $y, z \in A$, $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete and there is no cycle of l agents, $l \leq k$

Corollary

Let $\times_{i \in N} \mathcal{R}_i$ be a domain satisfying the k-size sequential inclusion condition. Then, any strategy-proof social choice function is k-group strategy-proof

SPECIAL RESULTS (4): Controlling for the alternatives in the range

Limitations on the size of the range:

Definition

A social choice function on a set A of alternatives and with range A_f is range based if and only if $f(R_N) = f(R'_N)$ whenever the restriction of R_N and R'_N to A_f are the same

Theorem

If a range-based social choice function f with range of size k is $(k-2)-$group strategy-proof, it is also group strategy-proof

SPECIAL RESULTS (4): Controlling for the alternatives in the range (cont.)

Limitations on the name of alternatives in the range: sequential inclusion may only hold for subsets of alternatives. If so, we can also guarantee the equivalence

Definition

A profile of preferences $R_N \in \times_{i \in N} \mathcal{R}_i$ satisfies sequential inclusion on $B \subset A$ if for any $y, z \in B$ the binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete and acyclic. A domain $\times_{i \in N} \mathcal{R}_i$ satisfies sequential inclusion on B if the condition holds for all profiles in it

Theorem

Let $\times_{i \in N} \mathcal{R}_i$ be a domain of preferences and B be a set of alternatives such that $\times_{i \in N} \mathcal{R}_i$ satisfies sequential inclusion on B. Then, any strategy-proof, range-based social choice function with range B is also group strategy-proof

SPECIAL RESULTS: (5) Partial result on necessity

Theorem

Let $\times_{i \in N} \mathcal{R}_i$ be a domain on which any strategy-proof social choice function on $\times_{i \in N} \mathcal{D}_i \subset \times_{i \in N} \mathcal{R}_i$ is also pairwise strategy-proof on $\times_{i \in N} \mathcal{D}_i$. Then, $\times_{i \in N} \mathcal{R}_i$ satisfies the 2-size sequential inclusion condition

Theorem

Let $\times_{i \in N} \mathcal{R}_i$ be a domain on which any strategy-proof social choice function on $\times_{i \in N} \mathcal{D}_i \subset \times_{i \in N} \mathcal{R}_i$ is also group strategy-proof on $\times_{i \in N} \mathcal{D}_i$. Then, $\times_{i \in N} \mathcal{R}_i$ satisfies the sequential inclusion condition

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